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Phase operator for the photon, an index theorem, and quantum anomaly †

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An index relation $dim\ ker\ a - dim\ ker\ a^\dagger = 1$ is satisfied by the creation and annihilation operators $a^\dagger$ and a of a harmonic oscillator. Implications of this analytic index on the possible form of the phase operator are discussed. A close analogy between the present phase operator problem and chiral anomaly in gauge theory, which is associated with Atiyah-Singer index theorem, is emphasized.

1. INDEX RELATIONS

The quantum phase operator has been studied by various authors in the past[1-4]. We here remark on the absence of a hermitian phase operator and the lack of a mathematical basis for $\Delta N \Delta\phi \geq 1/2$, on the basis of a notion of index or an index theorem. To be specific, we study the simplest one-dimensional harmonic oscillator defined by $h = a^\dagger a + 1/2$ where a and $a^\dagger$ stand for the annihilation and creation operators satisfying the standard commutator $[a, a^\dagger] = 1$. The vacuum state $|0\rangle$ is annihilated by a, $a|0\rangle = 0$ which ensures the absence of states with negative norm. The number operator defined by $N = a^\dagger a$ then has non-negative integers as eigenvalues, and the annihilation operator a is represented by

$$a = |0\rangle\langle 1| + |1\rangle\langle 2|\sqrt{2} + |2\rangle\langle 3|\sqrt{3} + ... \tag{1}$$

in terms of the eigenstates $|k\rangle$ of the number operator, $N|k\rangle = k|k\rangle$, with $k = 0, 1, 2, ...$ The creation operator $a^\dagger$ is given by the hermitian conjugate of a in (1).

In the representation of a and $a^\dagger$ specified above we have the index condition

$$dim\ ker\ a - dim\ ker\ a^\dagger = 1 \tag{2}$$

where $dim\ ker\ a$,for example, stands for the number of normalizable basis vectors u_n which satisfy $au_n = 0$.

In the conventional notation of index theory, the relation (2) is written by using the trace of well-defined operators as

$$Tr(e^{-a^\dagger a/M^2}) - Tr(e^{-aa^\dagger/M^2}) = 1 \tag{3}$$

The relation (3) is confirmed for the standard representation (1) as ,$1 + (\sum_{n=1}^{\infty} e^{-n/M^2}) - (\sum_{n=1}^{\infty} e^{-n/M^2}) = 1$, independently of the value of M^2:Eq.(3) may be taken as an altertnative definition of the index.

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If one should suppose the existence of a well defined hermitian phase operator ϕ, one would have a polar decomposition

$$a = U(\phi)H = e^{i\phi}H \tag{4}$$

as was originally suggested by Dirac[1]. Here U and H stand for unitary and hermitian operators, respectively. If (4) should be valid, we have in the same notation as (3)

$$Tr(e^{-a^{\dagger}a/M^2}) - Tr(e^{-aa^{\dagger}/M^2}) = Tr(e^{-H^2/M^2}) - Tr(e^{-UH^2U^{\dagger}/M^2}) = 0 \tag{5}$$

This relation when combined with (3) constitutes a proof of the absence of a hermitian phase operator in the framework of index theory.

The basic utility of the notion of index or an index theorem lies in the fact that the index as such is an integer and remains invariant under a wide class of continuous deformation. As an example, the unitary time development of a and $a^{\dagger}$ dictated by the Heisenberg equation of motion, which includes a fundamental phenomenon such as squeezing, does not alter the index relation.

If one truncates the representation of a to any finite dimension, for example, to an $(s+1)$ dimension with s a positive integer, one obtains by noting $a_s^{\dagger}|s\rangle = 0$

$$dim\ ker\ a_s - dim\ ker\ a_s^{\dagger} = 0 \tag{6}$$

where a_s stands for an $s+1$ dimensional truncation of a. This relation (6) is proved by noting that non-vanishing eigenvalues of $a_s^{\dagger}a_s$ and $a_s a_s^{\dagger}$ are in one-to-one correspondence: In the eigenvalue equations

$$a_s^{\dagger}a_s u_n = \lambda_n^2 u_n \tag{7}$$

one may define $v_n = a_s u_n/\lambda_n$ for $\lambda_n \neq 0$. One then obtains $a_s a_s^{\dagger} v_n = \lambda_n^2 v_n$ by multiplying a_s to both hand sides of eq. (7). For any finite dimensional matrix , one also has $Tr(a_s^{\dagger}a_s) = Tr(a_s a_s^{\dagger})$. These two facts combined then lead to the statement that $a_s^{\dagger}a_s$ and $a_s a_s^{\dagger}$ contain the same number of zero eigenvalues, which implies the relation (6) or

$$Tr_{(s+1)}(e^{-a_s^{\dagger}a_s/M^2}) - Tr_{(s+1)}(e^{-a_s a_s^{\dagger}/M^2}) = 0 \tag{8}$$

where $Tr_{(s+1)}$ denotes an $s+1$ dimensional trace. This index relation holds independently of M^2 and s.

2. PHASE OPERATORS

From the above analysis of the index condition , the polar decomposition (4) could be consistently defined if one truncates the dimension of the representation space to a finite $s+1$ dimension . This is the approach adopted by Pegg and Barnett[3] in their definition of a hermitian phase operator

$$e^{i\phi} = |0\rangle\langle 1| + |1\rangle\langle 2| + ... + |s-1\rangle\langle s| + e^{i(s+1)\phi_0}|s\rangle\langle 0| \tag{9}$$

where ϕ_0 is a constant c-number. One can also define hermitian functions

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}),\ \sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}) \tag{10}$$

with $e^{-i\phi} = (e^{i\phi})^\dagger$.These operators satisfy , among others,

$$[\cos\phi, \sin\phi] = 0,\ \cos^2\phi + \sin^2\phi = 1 \tag{11}$$

Their basic idea is then to let s arbitrarily large later. The kernels of a_s and $a_s^\dagger$ are given by

$$ker\ a_s = \{|0\rangle\},\quad ker\ a_s^\dagger = \{|s\rangle\} \tag{12}$$

The limit $s \rightarrow \infty$ of the relation (6) is thus singular since the kernels in (12) are ill-defined for $s \rightarrow \infty$.

On the other hand, a non-hermitian phase operator which faithfully reflects the index condition (2) was introduced by Susskind and Glogower[2]

$$e^{i\varphi} = \frac{1}{\sqrt{N+1}}a = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3| + ... \tag{13}$$

with $dim\ ker\ e^{i\varphi} - dim\ ker\ (e^{i\varphi})^\dagger = 1$. This $e^{i\varphi}$ is not unitary, as is witnessed by $(e^{i\varphi})^\dagger e^{i\varphi} = 1 - |0\rangle\langle 0|$ and $e^{i\varphi}(e^{i\varphi})^\dagger = 1$. One may still define hermitian operators

$$C(\varphi) = \frac{1}{2}(e^{i\varphi} + (e^{i\varphi})^\dagger),\ S(\varphi) = \frac{1}{2i}(e^{i\varphi} - (e^{i\varphi})^\dagger) \tag{14}$$

which, together with the number operator N, satisfy

$$[N, C(\varphi)] = -iS(\varphi),\ [N, S(\varphi)] = iC(\varphi)$$

$$[C(\varphi), S(\varphi)] = \frac{1}{2i}|0\rangle\langle 0|,\ C(\varphi)^2 + S(\varphi)^2 = 1 - \frac{1}{2}|0\rangle\langle 0| \tag{15}$$

The last two relations in (15) are "anomalous" whereas the relations in (11) are normal from a view point of classical-quantum analogy.

It was shown in ref. (5) that the apparently "anomalous" relations (15) are in fact more consistent with quantum phenomena in the sense that one can prove the uncertainty relations

$$\Delta N \Delta \sin\phi \geq \Delta N \Delta S(\varphi) \geq \frac{1}{2}|\langle p|C(\varphi)|p\rangle| = \frac{1}{2}|\langle p|\cos\phi|p\rangle|,$$

$$\Delta\cos\phi\Delta\sin\phi \geq \Delta C(\varphi)\Delta S(\varphi) \geq \frac{1}{4}|\langle p|0\rangle\langle 0|p\rangle| \geq 0 \tag{16}$$

for *any* physical state $|p\rangle$ which satisfy $\langle p|N^2|p\rangle < \infty$. The uncertainty for the relations in (11) is *always* larger than the uncertainty for the relations in (15). If a state $|p\rangle$ is a minimum uncertainty state for the set of variables $(N, \cos\phi, \sin\phi)$ in the sense that it gives rise to an equality in the uncertainty relation, the same state $|p\rangle$ is automatically the minimum uncertainty state for the set of variables $(N, C(\varphi), S(\varphi))$. But the other way around is not necessarily true in general. This discrepancy is caused by the state $|s\rangle$

in (9), which is also responsible for the index relation (12). The unitary operator $e^{i\phi}$ in (9) always carries a vanishing index

$$dim\ ker\ e^{i\phi} - dim\ ker\ (e^{i\phi})^{\dagger} = 0 \tag{17}$$

since the unitary operators simply relabel the basis vectors without changing the number of them. This index mismatch of (17) with (13),which is caused by the state $|s\rangle$, thus leads to a substantial deviation from minimum uncertainty in a characteristically quantum domain with small average photon numbers[5].The deviation from the minimum uncertainty becomes appreciable in a characteristically quantum domain since the non-vanishing matrix element, for example, $|\langle s|\sin\phi|p\rangle|^2$, for whatever large s may be, becomes more appreciable compared with other states $|n\rangle$ contributing to the matrix elements $\sum|\langle n|\sin\phi|p\rangle|^2$.

3. ANALOGY WITH CHIRAL ANOMALY

It was also emphasized in ref. (5) that this kind of phenomenon, i.e., an apprently anomalous behavior (15) is in fact more consistent with quantum phenomena is a close analogy of chiral anomaly in gauge theory, which is related to the Atiyah-Singer index theorem similar to (2). In the analysis of chiral anomaly in gauge theory, the mass of the Pauli-Villars regulator provides a cut-off parameter and one obtains a normal result expected on the basis of classical-quantum analogy for any finite cut-off parameter. But in the limit of the large regulator mass, the cut-off parameter does not decouple and induces an anomalous term which is dictated by the Atiyah-Singer index theorem.

Precisely the same phenomenon takes place in the phase operator problem if one identifies the parameter s as the cut-off parameter. One may rewrite the relation (8) as

$$Tr_{(s+1)}(e^{-a_s^{\dagger}a_s/M^2}) - Tr_{(s)}(e^{-a_s a_s^{\dagger}/M^2}) = Tr_{(s+1)}(e^{-a_s a_s^{\dagger}/M^2}) - Tr_{(s)}(e^{-a_s a_s^{\dagger}/M^2}) = 1 \tag{18}$$

where $Tr_{(s)}$ stands for the trace over the first s dimensional subspace of the $s+1$ dimensional space, and the right-hand side of (18) is the contribution of the state $|s\rangle$, i.e., the cut-off parameter. It can be confirmed that each term in the left-hand side of (18) has a well-defined limit for $s \to \infty$ and one recovers the non-trivial index relation (2), which in turn prohibits the existence of a hermitian phase operator. On the basis of this analysis, it was suggested in ref. (5) that the anomalous behavior in (15) is an inevitable and unavoidable quantum effect, not an artifact of our insufficient definition of phase operator.

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